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OSCILLATIONS OF HIGHER ORDER NEUTRAL DIFFERENTIAL EQUATIONS OF MIXED TYPE*

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ABSTRACT

In this paper, we establish some new oscillation theorems for neutral higher order functional differential equations of the form

$$\text{(E)} \qquad \frac{d^n}{dt^n}(x(t) + cx(t-h) + Cx(t+H)) + qx(t-g) + Qx(t+G) = 0,$$

where c, C, g, G, h and H are real constants, and q and Q are nonnegative real constants. The results of this paper improve noticeably the known oscillation theorems. By a new analysis technique we give weaker sufficient conditions for all solutions of equation (E) to be oscillatory.

1. Introduction

Consider the higher order neutral functional differential equations of the form

$$\text{(E)} \quad (x(t) + cx(t-h) + Cx(t+H))^{(n)} + qx(x-g) + Qx(t+G) = 0, \quad n \geq 1,$$

where c, C, g, G, h and H are real constants, and q and Q are nonnegative real constants. The oscillation theory of neutral differential equations has been extensively developed during the last few years. See, for example, [1–11] and the references cited therein. The purpose of this paper is to obtain some new easily verifiable sufficient conditions, involving the coefficients and the arguments only under which all solutions of (E) are oscillatory. Our technique, differing greatly from that in [5], here is based on the study of the characteristic equation

$$\text{(E}^*\text{)} \qquad \lambda^n(1 + ce^{-\lambda h} + Ce^{\lambda H}) + qe^{-\lambda g} + Qe^{\lambda G} = 0.$$

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Necessary and sufficient conditions (in terms of the characteristic equations) for the oscillations of all solutions of neutral differential equations have been established by Bilchev, Grammatikopoulos and Stavroulakis [1,2], Grove, Ladas and Meimaridou [6], Ladas, Partheniadis and Sficas [9], Sficas and Stavroulakis [10], and Wang [11].

The osillation criteria obtained in this paper improve noticeably the results in [4].

As is customary, a solution of equation (E) is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative. Equation (E) is called oscillatory if all its solutions are oscillatory.

2. Main results

The following Lemma, which will be used in the proofs of our oscillation results, is extracted from [1], [2] and [11].

LEMMA: *A necessary and sufficient condition for the oscillation of* (E) *is that its characteristic equation* (E*) *has no real roots.*

First, we study the oscillatory behavior of the mixed neutral differential equations of the form

$$(E_1) \qquad (x(t)+cx(t-h)-Cx(t+H))^{(n)}+qx(t-g)+Qx(t+G)=0,$$

$$(E_2) \qquad (x(t)-cx(t-h)+Cx(t+H))^{(n)}+qx(t-g)+Qx(t+G)=0,$$

$$(E_3) \qquad (x(t)+cx(t-h)-Cx(t-H))^{(n)}+qx(t-g)+Qx(t+G)=0,$$

and

$$(E_4) \qquad (x(t)+cx(t+h)-Cx(t+H))^{(n)}+qx(t-g)+Qx(t+G)=0,$$

where c, C, g, G, h and H are nonnegative real constants, and q and Q are positive real constants.

The characteristic equations of equations (E_1)–(E_4) are respectively

$$(E_1^*) \qquad F_1(\lambda):=\lambda^n(1+ce^{-\lambda h}-Ce^{\lambda H})+qe^{-\lambda g}+Qe^{\lambda G}=0,$$

$$(E_2^*) \qquad F_2(\lambda):=\lambda^n(1-ce^{-\lambda h}+Ce^{\lambda H})+qe^{-\lambda g}+Qe^{\lambda G}=0,$$

$$(E_3^*) \qquad F_3(\lambda):=\lambda^n(1+ce^{-\lambda h}-Ce^{-\lambda H})+qe^{-\lambda g}+Qe^{\lambda G}=0$$

and

$$(E_4^*) \qquad F_4(\lambda):=\lambda^n(1+ce^{\lambda h}-Ce^{\lambda H})+qe^{-\lambda g}+Qe^{\lambda G}=0.$$

THEOREM 1: *Assume that $C > 0$, $G > H$ and $g > h$,*

$$Q\left(\frac{e}{n}\right)^n (G-H)^n > C, \tag{1}$$

$$q\left(\frac{e}{n}\right)^n (g-h)^n > \exp\left[-\left(\frac{q}{1+c}\right)^{\frac{1}{n}} h\right] + c \quad \text{if } n \text{ is odd}, \tag{2}$$

and

(3)

$$q\left(\frac{e}{n}\right)^n (g+H)^n + \exp\left[\left(\frac{q}{C}\right)^{\frac{1}{n}} H\right] + c\exp\left[\left(\frac{q}{C}\right)^{\frac{1}{n}} (h+H)\right] > C \quad \text{if } n \text{ is even.}$$

Then (E_1) *is oscillatory.*

Proof: Our strategy is to prove that under the hypotheses above the characteristic equation (E_1^*) of (E) has no real roots in $(-\infty, \infty)$. There are three possible cases.

CASE 1: n is odd or even and $\lambda > 0$.

For $\lambda \neq 0$, we have

$$F_1(\lambda)e^{-\lambda H}/\lambda^n = \left(qe^{-\lambda(g+H)} + Qe^{\lambda(G-H)}\right)/\lambda^n + e^{-\lambda H} + ce^{-\lambda(h+H)} - C. \tag{4}$$

If $0 < \lambda \leq \left(\frac{Q}{C}\right)^{\frac{1}{n}}$, then from (4) we find

$$F_1(\lambda)e^{-\lambda H}/\lambda^n > Qe^{\lambda(G-H)}/\left(\frac{Q}{C}\right) - C > 0.$$

If $\lambda > \left(\frac{Q}{C}\right)^{\frac{1}{n}}$, then in view of (1) and using the inequality $e^x \geq ex$ $(x \geq 0)$, (4) yields

$$\begin{aligned} F_1(\lambda)e^{-\lambda H}/\lambda^n &> Q\left(e^{\frac{\lambda}{n}(G-H)}/\frac{\lambda}{n}(G-H)\right)^n \frac{1}{n^n}(G-H)^n - C \\ &\geq Q\left(\frac{e}{n}\right)^n (G-H)^n - C > 0. \end{aligned}$$

CASE 2: n is odd and $\lambda < 0$.

In this case we have

$$\begin{aligned} -F_1(\lambda)e^{\lambda h}/\lambda^n &= F_1(\lambda)e^{\lambda h}/(-\lambda)^n \\ &= \left(qe^{-\lambda(g-h)} + Qe^{\lambda(G+h)}\right)/(-\lambda)^n - \left(e^{\lambda h} + c - Ce^{l(H+h)}\right). \end{aligned} \tag{5}$$

If $-\left(\frac{q}{1+c}\right)^{\frac{1}{n}} \leq \lambda < 0$, then from (5) it follows that

$$-F_1(\lambda)e^{\lambda h}/\lambda^n > qe^{-\lambda(g-h)}/\left(\frac{q}{1+c}\right) - (1+c) > 0.$$

If $\lambda < -\left(\frac{q}{1+c}\right)^{\frac{1}{n}}$, then by using (2) and the inequality $e^x \geq ex$ $(x \geq 0)$, we obtain from (5) that

$$\begin{aligned} -F_1(\lambda)e^{\lambda h}/\lambda^n &= F_1(\lambda)e^{\lambda h}/(-\lambda)^n \\ &> q\left(\frac{e^{-\frac{\lambda}{n}(g-h)}}{-\frac{\lambda}{n}(g-h)}\right)^n \frac{1}{n^n}(g-h)^n - \left(e^{\lambda h} + c - Ce^{\lambda(H+h)}\right) \\ &\geq q\left(\frac{e}{n}\right)^n (g-h)^n - \exp\left[-\left(\frac{q}{1+c}\right)^{\frac{1}{n}} h\right] - c > 0. \end{aligned}$$

CASE 3: n is even and $\lambda < 0$.

If $-\left(\frac{q}{C}\right)^{\frac{1}{n}} \leq \lambda < 0$, then by (4) it follows that

$$F_1(\lambda)e^{-\lambda H}/\lambda^n > qe^{-\lambda(g+H)}/\left(\frac{q}{C}\right) - C > 0.$$

If $\lambda < -\left(\frac{q}{c}\right)^{\frac{1}{n}}$, then by using (3) we obtain

$$F_1(\lambda)e^{-\lambda H}/\lambda^n \geq q\left(\frac{e}{n}\right)^n (g+H)^n + e^{(\frac{q}{c})H} + ce^{(\frac{q}{C})(h+H)} - C > 0.$$

Cases 1–3 and $F_1(0) > 0$ imply that $F_1(\lambda) > 0$ for $\lambda \in (-\infty, \infty)$, that is, (E_1^*) has no real roots. By Lemma we conclude that (E_1) is oscillatory. The proof of the theorem is complete. ∎

THEOREM 2: *Assume that $c > 0$ and $g > h$,*

$$(6)\qquad Q\left(\frac{e}{n}\right)^n (G+h)^n + \exp\left[\left(\frac{Q}{c}\right)^{\frac{1}{n}} h\right] + C\cdot\exp\left[\left(\frac{Q}{c}\right)^{\frac{1}{n}} (H+h)\right] > c,$$

(7)

$$q\left(\frac{e}{n}\right)^n g^n + c\cdot\exp\left[\left(\frac{q}{1+c}\right)^{\frac{1}{n}} h\right] > 1 + C\cdot\exp\left[-\left(\frac{q}{1+C}\right)^{\frac{1}{n}} H\right] \quad \textit{if } n \textit{ is odd},$$

and

$$(8)\qquad q\left(\frac{e}{n}\right)^n (g-h)^n > c \quad \textit{if } n \textit{ is even}.$$

Then (E_2) *is oscillatory.*

Proof: For $\lambda \neq 0$, we have

$$(9) \quad F_2(\lambda)e^{\lambda h}/\lambda^n = \left(qe^{-\lambda(g-h)} + Qe^{\lambda(G+h)}\right)/\lambda^n + \left(e^{\lambda h} - c + Ce^{\lambda(H+h)}\right).$$

Now, we consider the following cases.

CASE 1: n is odd or even and $\lambda > 0$.

If $0 < \lambda \leq \left(\frac{Q}{c}\right)^{\frac{1}{n}}$, then from (9) we find

$$F_2(\lambda)e^{\lambda h}/\lambda^n > Qe^{\lambda(G+H)}/\left(\frac{Q}{c}\right) - c > 0.$$

If $\lambda > \left(\frac{Q}{c}\right)^{\frac{1}{n}}$, then from (9) it is easy to see that

$$F_2(\lambda)e^{\lambda h}/\lambda^n > Q\left(\frac{e}{n}\right)^n (G+h)^n + \exp\left[\left(\frac{Q}{c}\right)^{\frac{1}{n}} h\right] - c + C \cdot \exp\left[\left(\frac{Q}{c}\right)^{\frac{1}{n}} (H+h)\right] > 0.$$

CASE 2: n is odd and $\lambda < 0$.

In this case we have

$$(10) \quad \begin{aligned} -F_2(\lambda)/\lambda^n &= F_2(\lambda)/(-\lambda)^n \\ &= \left(qe^{-\lambda g} + Qe^{\lambda G}\right)/(-\lambda)^n - \left(1 - ce^{-\lambda h} + Ce^{\lambda H}\right). \end{aligned}$$

If $-\left(\frac{q}{1+C}\right) \leq \lambda < 0$, then in view of (10) we find

$$-F_2(\lambda)/\lambda^n > qe^{-\lambda g}/\left(\frac{q}{1+C}\right) - 1 - Ce^{\lambda H} > 0.$$

If $\lambda < -\left(\frac{q}{1+C}\right)^{\frac{1}{n}}$, then by (10) and (7)

$$-F_2(\lambda)/\lambda^n > q\left(e^{-\frac{\lambda}{n}g}/-\frac{\lambda}{n}g\right)^n \frac{1}{n^n}g^n - 1 + c \cdot \exp\left[\left(\frac{q}{1+C}\right)^{\frac{1}{n}} h\right] - C \cdot \exp\left[-\left(\frac{q}{1+C}\right)^{\frac{1}{n}} H\right] > 0.$$

CASE 3: n is even and $\lambda < 0$.

If $-\left(\frac{q}{c}\right)^{\frac{1}{n}} \le \lambda < 0$, from (9) one can easily see that

$$F_2(\lambda)e^{\lambda h}/\lambda^n > qe^{-\lambda(g-h)}/\left(\frac{q}{c}\right) + e^{\lambda h} - c + Ce^{\lambda(H+h)} > 0.$$

If $\lambda < -\left(\frac{q}{c}\right)^{\frac{1}{n}}$, then by using (9) and (8) we obtain that

$$F_2(\lambda)e^{\lambda h}/\lambda^n \ge q\left(\frac{e}{n}\right)^n (g-h)^n - c > 0.$$

From cases 1–3 and $F_2(0) > 0$ we can conclude that $F_2(\lambda) > 0$ for $\lambda \in (-\infty, \infty)$, that is, ($E_2^*$) has no real roots. So the conclusion of the theorem follows by applying the Lemma.

COROLLARY 1: *Let n be odd, $0 < c \le 1$ and condition (7) hold. Then* (E_2) *is oscillatory.*

Proof: Assume $\lambda \ge 0$. Since n is odd and $0 < c \le 1$, it follows that $F_2(\lambda) > 0$. Assume $\lambda < 0$; then by the procedure of the proof of Theorem 2 we see that $F_2(\lambda) > 0$. By applying Lemma we can complete the proof. ∎

The following two theorems provide sufficient conditions for equations (E_3) and (E_4) to be oscillatory. The proofs are similar to that of Theorems 1 and 2. Hence we omit the details.

THEOREM 3: *Assume that $C > 0$ and $g > H$. Moreover, suppose that*

$$Q\left(\frac{e}{n}\right)^n (G+H)^n + \exp\left[\left(\frac{Q}{C}\right)^{\frac{1}{n}} H\right] > C, \tag{11}$$

$$q\left(\frac{e}{n}\right)^n (g-h)^n > 1 + c, \quad \textit{if } n \textit{ is odd}, \tag{12}$$

and

$$q\left(\frac{e}{n}\right)^n (g-H)^n + c\cdot\exp\left[\left(\frac{q}{C}\right)^{\frac{1}{n}} (h-H)\right] > C, \quad \textit{if } n \textit{ is even}. \tag{13}$$

Then (E_3) *is oscillatory.*

We can obtain the following corollary similar to Corollary 1 for (E_3) with $0 < C \le 1$ and n odd. Here we omit the details.

COROLLARY 2: *Let n be odd, $0 < C \leq 1$ and condition (12) hold. Then* (E_3) *is oscillatory.*

THEOREM 4: *Assume that $C > 0$, $G > H$ and condition (1) holds. Moreover, suppose that*

$$q\left(\frac{e}{n}\right)^n g^n > 1 + c\cdot\exp\left[-\left(\frac{q}{1+c}\right)^{\frac{1}{n}} h\right], \quad \text{if } n \text{ is odd,} \tag{14}$$

and

$$q\left(\frac{e}{n}\right)^n (g+H)^n + \exp\left[\left(\frac{q}{C}\right)^{\frac{1}{n}} H\right] > C, \quad \text{if } n \text{ is even.} \tag{15}$$

Then (E_4) *is oscillatory.*

Next, we consider the neutral differential equations

$$(x(t) + cx(t-h) + Cx(t+H))^{(n)} + qx(t-g) + Qx(t+G) = 0, \tag{E$_5$}$$

$$(x(t) + cx(t-h) + Cx(t-H))^{(n)} + qx(t-g) + Qx(t+G) = 0, \tag{E$_6$}$$

and

$$(x(t) + cx(t+h) + Cx(t+H))^{(n)} + qx(t-g) + Qx(t+G) = 0, \tag{E$_7$}$$

where c, C and Q are nonnegative real constants and g, G, h, H and q are positive real constants. The characteristic equations of (E_5)–(E_7) are respectively

$$F_5(\lambda) := \lambda^n(1 + ce^{-\lambda h} + Ce^{\lambda H}) + qe^{-\lambda g} + Qe^{\lambda G} = 0, \tag{E$_5^*$}$$

$$F_6(\lambda) := \lambda^n(1 + ce^{-\lambda h} + Ce^{-\lambda H}) + qe^{-\lambda g} + Qe^{\lambda G} = 0, \tag{E$_6^*$}$$

and

$$F_7(\lambda) := \lambda^n(1 + ce^{\lambda h} + Ce^{\lambda H}) + qe^{-\lambda g} + Qe^{\lambda G} = 0. \tag{E$_7^*$}$$

THEOREM 5: *Suppose n is odd and $g > h$. Moreover, assume that*

$$q\left(\frac{e}{n}\right)^n (g-h)^n > \exp\left[-\left(\frac{q}{1+c+C}\right)^{\frac{1}{n}} h\right] + c + C\cdot\exp\left[-\left(\frac{q}{1+c+C}\right)^{\frac{1}{n}} (H+h)\right]. \tag{16}$$

Then (E_5) *is oscillatory.*

Proof: Since n is odd, it follows that $F_5(\lambda) > 0$ for $\lambda \geq 0$. Hence it remains to prove that $F_5(\lambda) > 0$ holds also for $\lambda < 0$. Indeed,

$$-F_5(\lambda)e^{\lambda h}/\lambda^n = \frac{1}{(-\lambda)^n}\left(qe^{-\lambda(g-h)} + Qe^{\lambda(G+h)}\right) - \left(e^{\lambda h} + c + Ce^{\lambda(H+h)}\right). \tag{17}$$

If $-\left(\frac{q}{1+c+C}\right) \leq \lambda < 0$, then by using (17) we observe that

$$-F_5(\lambda)e^{\lambda h}/\lambda^n > qe^{-\lambda(g-h)}/\frac{q}{1+c+C} - (1+c+C) > 0.$$

If $\lambda < -\left(\frac{q}{1+c+C}\right)^{\frac{1}{n}}$, then from (16) and (17) we have

$$-F_5(\lambda)e^{\lambda h}/\lambda^n > q\left(\frac{e}{n}\right)^n (g-h)^n - \exp\left[-\left(\frac{q}{1+c+C}\right)^{\frac{1}{n}} h\right] - c - C\exp\left[-\left(\frac{q}{1+c+C}\right)(H+h)\right] > 0.$$

Thus $F_5(\lambda) > 0$ for all $\lambda \in (-\infty, \infty)$, that is, (E_5^*) has no real roots. So the conclusion of the theorem follows by the Lemma. ∎

The following two theorems can be proved in a similar way and hence we omit the details.

THEOREM 6: *If n is odd, $g > h \geq H$ and*

$$q\left(\frac{e}{n}\right)^n (g-h)^n > \exp\left[-\left(\frac{q}{1+c+C}\right)^{\frac{1}{n}} h\right] + c + C\cdot\exp\left[-\left(\frac{q}{1+c+C}\right)^{\frac{1}{n}} (h-H)\right],$$

then (E_6) *is oscillatory.*

THEOREM 7: *Assume that n is odd and*

$$q\left(\frac{e}{n}\right)^n g^n > 1 + c\exp\left[-\left(\frac{q}{1+c+C}\right)^{\frac{1}{n}} h\right] + C\exp\left[-\left(\frac{q}{1+c+C}\right)^{\frac{1}{n}} H\right],$$

then (E_7) *is oscillatory.*

Finally, we consider the following neutral functional differential equations:

$$(E_8) \qquad (x(t) - cx(t-h) - Cx(t+H))^{(n)} + qx(t-g) + Qx(t+G) = 0,$$

$$(E_9) \qquad (x(t) - cx(t-h) - Cx(t-H))^{(n)} + qx(t-g) + Qx(t+G) = 0,$$

and

$$(E_{10}) \qquad (x(t) - cx(t+h) - Cx(t+H))^{(n)} + qx(t-g) + Qx(t+G) = 0,$$

where c, C, h and H are nonnegative real constants, g, G, q and Q are positive constants and $c + C > 0$. The characteristic equations of (E_8)–(E_{10}) are respectively

$$(E_8^*) \qquad F_8(\lambda) := \lambda^n\left(1 - ce^{-\lambda h} - Ce^{\lambda H}\right) + qe^{-\lambda g} + Qe^{\lambda G} = 0,$$

$$(E_9^*) \qquad F_9(\lambda) := \lambda^n\left(1 - ce^{-\lambda h} - Ce^{-\lambda H}\right) + qe^{-\lambda g} + Qe^{\lambda G} = 0,$$

and

$$(E_{10}^*) \qquad F_{10}(\lambda) := \lambda^n\left(1 - ce^{\lambda h} - Ce^{\lambda H}\right) + qe^{-\lambda g} + Qe^{\lambda G} = 0.$$

We establish the oscillation criteria for (E_8)–(E_{10}).

THEOREM 8: *Suppose that $g > h$ and $G > H$,*

$$(18) \qquad Q\left(\frac{e}{n}\right)^n (G-H)^n > c \cdot \exp\left[-\left(\frac{Q}{c+C}\right)^{\frac{1}{n}} (H+h)\right] + C,$$

$$(19) \qquad q\left(\frac{e}{n}\right)^n (g-h)^n > c + C \cdot \exp\left[-\left(\frac{q}{c+C}\right)^{\frac{1}{n}} (H+h)\right] \quad \text{if } n \text{ is even},$$

and

$$(20) \qquad q\left(\frac{e}{n}\right)^n g^n + c \cdot \exp\left(q^{\frac{1}{n}} h\right) > 1 \quad \text{if } n \text{ is odd}.$$

Then (E_8) *is oscillatory.*

THEOREM 9: *Suppose that $g > h \geq H$,*

$$Q\left(\frac{e}{n}\right)^n (G+H)^n + \exp\left[\left(\frac{Q}{c+C}\right)^{\frac{1}{n}} H\right] - c\exp\left[-\left(\frac{Q}{c+C}\right)^{\frac{1}{n}}(h-H)\right] - C > 0,$$

$$q\left(\frac{e}{n}\right)^n (g-h)^n > c + C\cdot\exp\left[-\left(\frac{q}{c+C}\right)^{\frac{1}{n}}(h-H)\right] \quad \text{if } n \text{ is even,}$$

and

$$q\left(\frac{e}{n}\right)^n g + C\cdot\exp\left(-q^{\frac{1}{n}}H\right) > 1 \quad \text{if } n \text{ is odd.}$$

Then (E_9) *is oscillatory.*

THEOREM 10: *Suppose that $G > H \geq h$,*

$$Q\left(\frac{e}{n}\right)^n (G-H)^n > c\cdot\exp\left[-\left(\frac{Q}{c+C}\right)^{\frac{1}{n}}(H-h)\right] + C > 0,$$

$$q\left(\frac{e}{n}\right)^n (g+h)^n + \exp\left[\left(\frac{q}{c+C}\right)^{\frac{1}{n}} h\right]$$

$$> c + C\cdot\exp\left[-\left(\frac{q}{c+C}\right)^{\frac{1}{n}}(H-h)\right] \quad \text{if } n \text{ is even,}$$

and

$$q\left(\frac{e}{n}\right)^n g^n > 1 \quad \text{if } n \text{ is odd.}$$

Then (E_{10}) *is oscillatory.*

Proof: We present the proof of Theorem 8. The proofs of Theorems 9 and 10 can be treated in a similar way. For $\lambda \neq 0$, we have

$$\text{(21)} \quad F_8(\lambda)e^{-\lambda H}/\lambda^n = \left(qe^{-\lambda(g+H)} + Qe^{\lambda(G-H)}\right)/\lambda^n + e^{-\lambda H} - ce^{-\lambda(h+H)} - C.$$

CASE 1: n is even or odd and $\lambda > 0$.

If $0 < \lambda \leq \left(\frac{Q}{c+C}\right)^{\frac{1}{n}}$, then from (21) we have

$$F_8(\lambda)e^{-\lambda H}/\lambda^n > Q/\left(\frac{Q}{c+C}\right) + e^{-\lambda H} - c - C > 0.$$

If $\lambda > \left(\frac{Q}{c+C}\right)^{\frac{1}{n}}$, then in view of (21) and (18) we find that

$$F_8(\lambda)e^{-\lambda H}/\lambda^n \geq Q\left(\frac{e}{n}\right)^n (G-H)^n - c\exp\left[-\left(\frac{Q}{c+C}\right)^{\frac{1}{n}}(h+H)\right] - C > 0.$$

CASE 2: n is even and $\lambda < 0$.

In this case we have

$$F_8(\lambda)e^{\lambda h}/\lambda^n = \left(qe^{-\lambda(g-h)} + Qe^{\lambda(G+h)}\right)/(-\lambda)^n + \left(e^{\lambda h} - c - Ce^{\lambda(H+h)}\right).$$

If $-\left(\frac{q}{c+C}\right)^{\frac{1}{n}} \le \lambda < 0$, then

$$F_8(\lambda)e^{\lambda h}/\lambda^n > q/\left(\frac{q}{c+C}\right) + e^{\lambda h} - c - C > 0.$$

If $\lambda < -\left(\frac{q}{c+C}\right)^{\frac{1}{n}}$, then from (21) and (19) we obtain that

$$F_8(\lambda)e^{\lambda h}/\lambda^n \ge q\left(\frac{e}{n}\right)^n (g-h)^n - c - C\exp\left[-\left(\frac{q}{c+C}\right)^{\frac{1}{n}}(H+h)\right] > 0.$$

CASE 3: n is odd and $\lambda < 0$.

In this case we have

$$-F_8(\lambda)/\lambda^n = F_8(\lambda)/(-\lambda)^n = \left(qe^{-\lambda g} + Qe^{\lambda G}\right)/(-\lambda)^n - \left(1 - ce^{-\lambda h} - Ce^{\lambda H}\right).$$

If $-q^{\frac{1}{n}} \le \lambda < 0$, then by using (21) we obtain that

$$-F_8(\lambda)/\lambda^n > q/(q-1) + c + C > 0.$$

If $\lambda < -q^{\frac{1}{n}}$, then from (20) it follows that

$$-F_8(\lambda)/\lambda^n \ge q\left(\frac{e}{n}\right)^n g^n - 1 + c\cdot\exp\left(q^{\frac{1}{n}}h\right) > 0.$$

Cases 1–3 and $F_8(0) > 0$ imply $F_8(\lambda) > 0$ for $\lambda \in (-\infty, \infty)$, that is, ($E_8^*$) has no real roots. By using the Lemma, we prove that (E_8) is oscillatory. The proof is complete. ∎

Remarks: 1. It is easy to see that Theorems 1–10 improve respectively Theorems 1–10 in [4].

2. By using the technique of this paper we can obtain more oscillation criteria for equations (E_1) – (E_{10}).

3. Our technique can be extended to a higher order neutral equation of the form

$$\left(x(t) + \sum_{i=1}^{n_1} c_i x(t-h_i) + \sum_{j=1}^{n_2} C_j x(t+H_j)\right)^{(n)} \pm \left(\sum_{k=1}^{n_3} q_k x(t-g_k) + \sum_{m=1}^{n_4} Q_m x(t+G_m)\right) = 0, \quad n \ge 1,$$

where c_i, C_j, h_i and H_j are real constants, q_k, g_k, Q_m and G_m are nonnegative real constants.

4. It is easy to construct examples showing that our criteria for oscillation of $(\mathrm{E}_1) - (\mathrm{E}_{10})$ are essentially wider than the oscillation criteria in [4].

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